SOLUTION OF A SYSTEM OF EQUATIONS OCCURRING IN DARBOUX'S *THÉORIE GÉNÉRALE DES SURFACES*.*

By Dr. T. Craig, Baltimore, Md.

The equations are, to notation *près*,

$$\text{(A)}\qquad \begin{aligned} x^2 + y^2 + z^2 &= \text{const.} \\ xx_1 + yy_1 + zz_1 &= \text{const.} \\ xx_2 + yy_2 + zz_2 &= \text{const.}, \end{aligned}$$

and we have also

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= \text{const.} \\ x_2^2 + y_2^2 + z_2^2 &= \text{const.} \end{aligned}$$

It is required to find x, y, z from the first three of these. Darboux gives as the solution, without any indication of the process employed, the following:

$$\begin{aligned} x &= A_1x_1 + A_2x_2 + A_3(y_1z_2 - y_2z_1), \\ y &= A_1y_1 + A_2y_2 + A_3(z_1x_2 - z_2x_1), \\ z &= A_1z_1 + A_2z_2 + A_3(x_1y_2 - x_2y_1). \end{aligned}$$

A method for the solution of a general system of equations of the form (A) exists due to Bauer, but for the present case the following direct method, which is based on the most elementary geometrical considerations, seems to me preferable. The geometrical interpretation of the results and the notation is so obvious that it need not be referred to.

Write the equations in the form

$$\begin{aligned} x^2 + y^2 + z^2 &= C^2, \\ x_1^2 + y_1^2 + z_1^2 &= C_1^2, \\ x_2^2 + y_2^2 + z_2^2 &= C_2^2, \end{aligned} \qquad (1)$$

$$\begin{aligned} xx_1 + yy_1 + zz_1 &= CC_1\lambda_1, \\ xx_2 + yy_2 + zz_2 &= CC_2\lambda_2; \end{aligned} \qquad (2)$$

where C, C_1, C_2 are arbitrary constants. Make now

$$\begin{aligned} x, y, z &= C\alpha, C\beta, C\gamma, \\ x_1, y_1, z_1 &= C_1\alpha_1, C_1\beta_1, C_1\gamma_1, \\ x_2, y_2, z_2 &= C_2\alpha_2, C_2\beta_2, C_2\gamma_2. \end{aligned} \qquad (3)$$

* t. I, p. 21; the equations preceding (6).

The three equations which we have ultimately to solve are now

$$\left.\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 1 \\ \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 &= \lambda_1 \\ \alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2 &= \lambda_2 \end{aligned}\right\} \qquad (4)$$

with

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$$

$$\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1 .$$

Define three new quantities x_3, y_3, z_3 by the equations

$$\left.\begin{aligned} x_3^2 + y_3^2 + z_3^2 &= C_3^2 , \\ x_1x_3 + y_1y_3 + z_1z_3 &= 0 , \\ x_2x_3 + y_2y_3 + z_2z_3 &= 0 ; \end{aligned}\right\} \qquad (5)$$

also write

$$xx_3 + yy_3 + zz_3 = CC_3\lambda_3'$$

Making

$$x_3,\ y_3,\ z_3 = C_3\alpha_3,\ C_3\beta_3,\ C_3\gamma_3$$

we have

$$\left.\begin{aligned} \alpha_3^2 + \beta_3^2 + \gamma_3^2 &= 1 , \\ \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 &= 0 , \\ \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 &= 0 , \\ \alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3 &= \lambda_3' . \end{aligned}\right\} \qquad (6)$$

The second and third of these give

$$\frac{\alpha_3}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\beta_3}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\gamma_3}{\alpha_1\beta_2 - \alpha_2\beta_1} . \qquad (7)$$

Write

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = \cos\theta ; \qquad (8)$$

square the terms in (7), add the numerators together and the denominators together, use the first of (6) and extract the square root of the result, this will be the common value of the ratios in (6), viz:

$$\frac{\alpha_3}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\beta_3}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\gamma_3}{\alpha_1\beta_2 - \alpha_2\beta_1} = \frac{1}{\sin\theta} , \qquad (9)$$

or

$$\alpha_3 = \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\sin\theta} , \quad \beta_3 = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\sin\theta} , \quad \gamma_3 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\sin\theta} . \qquad (10)$$

These satisfy identically the first three of equations (6). Take now the last two of equations (4) and the last of (6); these are

$$\begin{aligned} \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 &= \lambda_1\,, \\ \alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2 &= \lambda_2\,, \\ \alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3 &= \lambda_3\,. \end{aligned} \tag{11}$$

Substituting the above values of α_3, β_3, γ_3, these become

$$\begin{aligned} \alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) &= \lambda_3' \sin\theta = \lambda_3\,, \\ \alpha\alpha_1 \qquad + \beta\beta_1 \qquad + \gamma\gamma_1 \qquad &= \lambda_1\,, \\ \alpha\alpha_2 \qquad + \beta\beta_2 \qquad + \gamma\gamma_2 \qquad &= \lambda_2\,. \end{aligned} \tag{12}$$

Solving these we find

$$\begin{aligned} \alpha &= \frac{\lambda_3}{\sin^2\theta}(\beta_1\gamma_2 - \beta_2\gamma_1) + \frac{\lambda_2}{\sin^2\theta}(\alpha_2 - \alpha_1\cos\theta) + \frac{\lambda_1}{\sin^2\theta}(\alpha_1 - \alpha_2\cos\theta)\,, \\ \beta &= \frac{\lambda_3}{\sin^2\theta}(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \frac{\lambda_2}{\sin^2\theta}(\beta_2 - \beta_1\cos\theta) + \frac{\lambda_1}{\sin^2\theta}(\beta_1 - \beta_2\cos\theta)\,, \\ \gamma &= \frac{\lambda_3}{\sin^2\theta}(\alpha_1\beta_2 - \alpha_2\beta_1) + \frac{\lambda_2}{\sin^2\theta}(\gamma_2 - \gamma_1\cos\theta) + \frac{\lambda_1}{\sin^2\theta}(\gamma_1 - \gamma_2\cos\theta)\,. \end{aligned} \tag{13}$$

Write

$$\frac{\lambda_1}{\sin^2\theta} - \frac{\lambda_2\cos\theta}{\sin^2\theta} = l_1\,, \quad \lambda_1 = l_1 + l_2\cos\theta\,,$$

$$-\frac{\lambda_1\cos\theta}{\sin^2\theta} + \frac{\lambda_2}{\sin^2\theta} = l_2\,, \quad \lambda_2 = l_2 + l_1\cos\theta\,.$$

Equations (13) now become

$$\begin{aligned} \alpha &= l_1\alpha_1 + l_2\alpha_2 + \frac{\lambda_3}{\sin^2\theta}(\beta_1\gamma_2 - \beta_2\gamma_1)\,, \\ \beta &= l_1\beta_1 + l_2\beta_2 + \frac{\lambda_3}{\sin^2\theta}(\gamma_1\alpha_2 - \gamma_2\alpha_1)\,, \\ \gamma &= l_1\gamma_1 + l_2\gamma_2 + \frac{\lambda_3}{\sin^2\theta}(\alpha_1\beta_2 - \alpha_2\beta_1)\,. \end{aligned} \tag{14}$$

To determine λ_3 (which with the auxiliaries α_3, β_3, γ_3 is not arbitrary) we use the first of (4), viz.,

$$\alpha^2 + \beta^2 + \gamma^2 = 1\,. \tag{15}$$

This gives at once

$$\frac{\lambda_3}{\sin\theta} = \frac{\sqrt{1 - l_1^2 - l_2^2 - 2l_1 l_2 \cos\theta}}{\sin\theta} =, \text{ say, } l_3 . \tag{16}$$

So we have now

$$\begin{aligned} \alpha &= l_1\alpha_1 + l_2\alpha_2 + l_3(\beta_1\gamma_2 - \beta_2\gamma_1), \\ \beta &= l_1\beta_1 + l_2\beta_2 + l_3(\gamma_1\alpha_2 - \gamma_2\alpha_1), \\ \gamma &= l_1\gamma_1 + l_2\gamma_2 + l_3(\alpha_1\beta_2 - \alpha_2\beta_1). \end{aligned} \tag{17}$$

These are of the required form ; to get back to x, y, z write

$$l_1,\ l_2,\ l_3, = C_1 m_1,\ C_2 m_2,\ C_1 C_2 m_3 ;$$

then multiply each of (17) through by the arbitrary constant C and again write

$$Cm_1,\ Cm_2,\ Cm_3 = A_1,\ A_2,\ A_3,$$

where A_1, A_2, A_3 are arbitrary constants, and we have finally

$$\begin{aligned} x &= A_1 x_1 + A_2 x_2 + A_3(y_1 z_2 - y_2 z_1), \\ y &= A_1 y_1 + A_2 y_2 + A_3(z_1 x_2 - z_2 x_1), \\ z &= A_1 z_1 + A_2 z_2 + A_3(x_1 y_2 - x_2 y_1), \end{aligned}$$

the required values. A number of modifications of the preceding process naturally suggest themselves, but it is not worth while to go into them.

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